# Higher-Order Inclusions of Nonlinear Systems by Chebyshev Models

Jai Rajyaguru[1], Mario E. Villanueva[1], Boris Houska[2] and Benoît Chachuat[1]

(1) Centre for Process Systems Engineering, Department of Chemical Engineering, Imperial College London, UK
(2) School of Information Science and Technology, ShanghaiTech University, 319 Yueyang Road, Shanghai 200031, China

## Introduction

**Aim:**
Develop a method to create an approximating polynomial for a function which minimises the maximum error at any point. As well as being able to obtain an guaranteed bound on this error.

**Definition:**
Given some function $f(x)$ for which we create a polynomial approximation $\mathcal{P}(x)$, over a given variable range $x_l \leq x \leq x_u$ we can call the maximum approximation error $\mathcal{E}$:

$$\mathcal{E} = \max_{x_l \leq x \leq x_u} |f(x) - \mathcal{P}(x)|$$

We would like to develop a method for generating $\mathcal{P}(x)$, which also allows us to calculate a bound for the error.

## Approach

We create the polynomial approximation and propagate the error bound stepwise using the factorable function operations.

**Addition:** $\mathcal{P}_1(x) + \mathcal{P}_2(x)$

1. Add the polynomials
2. Add the error bounds

**Multiplication:** $\mathcal{P}_1(x)\mathcal{P}_2(x)$

1. Multiply the polynomials
2. Reduce the order by bounding higher order terms
3. Multiply the error bounds

**Univariate:** $\mathcal{U}(P_1(x))$

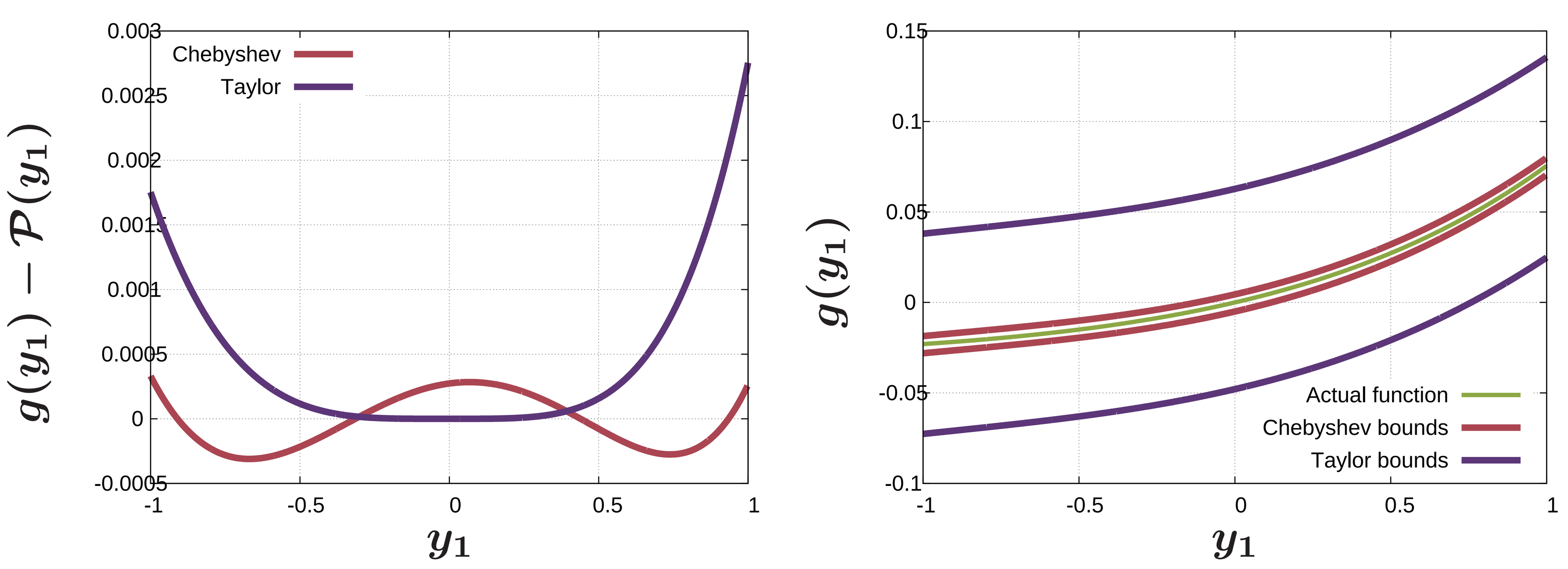
1. Bound the range of the inner polynomial
2. Create a polynomial approximation for the univariate function
3. Find the approximation error for this expansion
4. Compose the inner polynomial into the expansion
5. Reduce the order by bounding higher order terms
6. Propagate the error bound from the inner polynomial

## Example - Factorable Function

$$g(y_1, y_2) = \frac{y_1 y_2 \exp(y_1)}{(y_1+5)^2}$$

$$-1 \leq y_1 \leq 1$$
$$y_2 = 1$$

Polynomial approximation order 3




- The Taylor approximation is only exact at one point while the Chebyshev approximation is at several.
- Chebyshev bounds obtained by adding and subtracting the error from the expansion is significantly better than the Taylor equivalent.

## Conclusions

- Chebyshev expansions provide better approximations for factorable functions.
- This effect becomes amplified when propagating error bounds as with every operation you get a reduction in overestimation.

## Factorable Functions

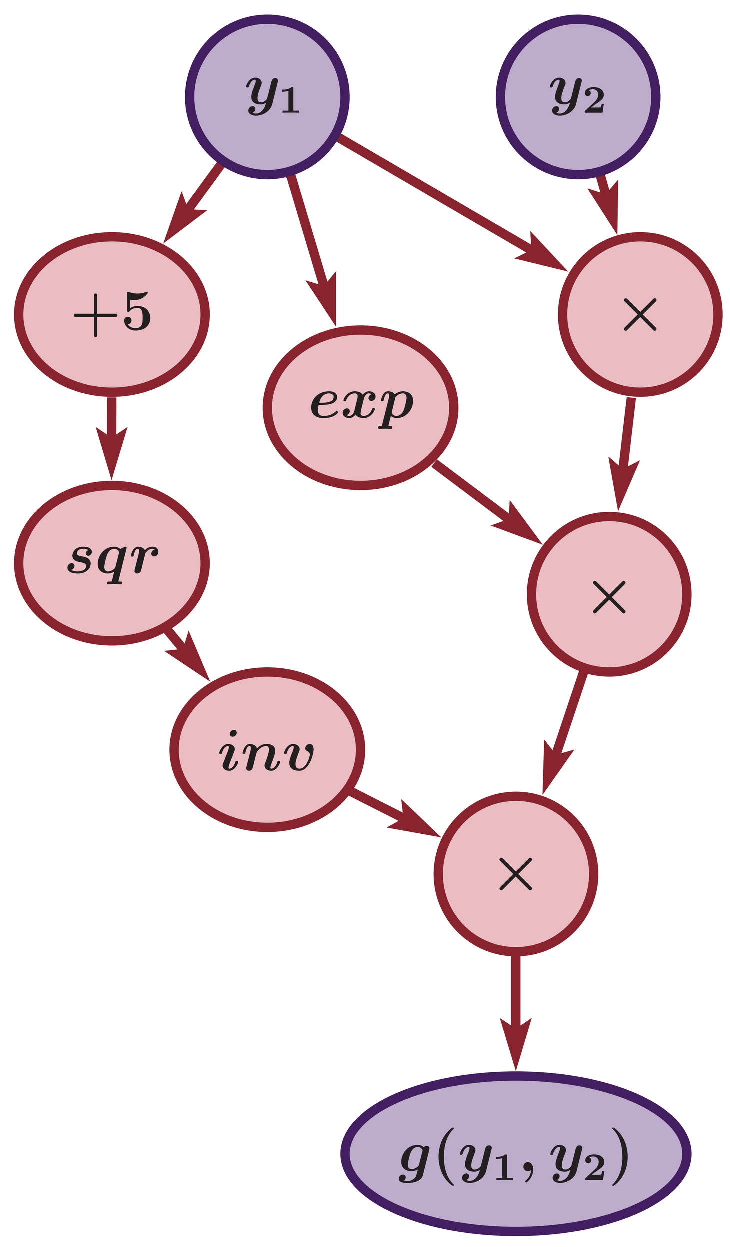
**Definition:**
A function that can be broken into a finite number of bivariate additions, multiplications and univariate operations.

**Exceptions:**
Not all solutions have a factorable form, however there are methods that can be used to obtain them with factorable operations.

- Differential equations using Euler method
- Implicit equations using Newton method

**Example:**

$$g(y_1, y_2) = \frac{y_1 y_2 \exp(y_1)}{(y_1+5)^2}$$




## Taylor

- The basis of this expansion are monomials defined as $x^n$.
- Coefficients are calculated by finding derivatives at a single point, $x_0$.

$$\mathcal{P}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots$$

## Chebyshev

- The basis of this expansion are Chebyshev polynomials defined as $T_n = \cos(n \arccos(x))$.
- Coefficients are calculated via integration over a range.

$$\mathcal{P}(x) = \frac{a_0}{2} + a_1 T_1(x) + a_2 T_2(x) + \ldots$$

where

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1-x^2}} dx$$

## Example - Differential Equation

Simplified Lotka-Volterra equations (Predator-Prey model):

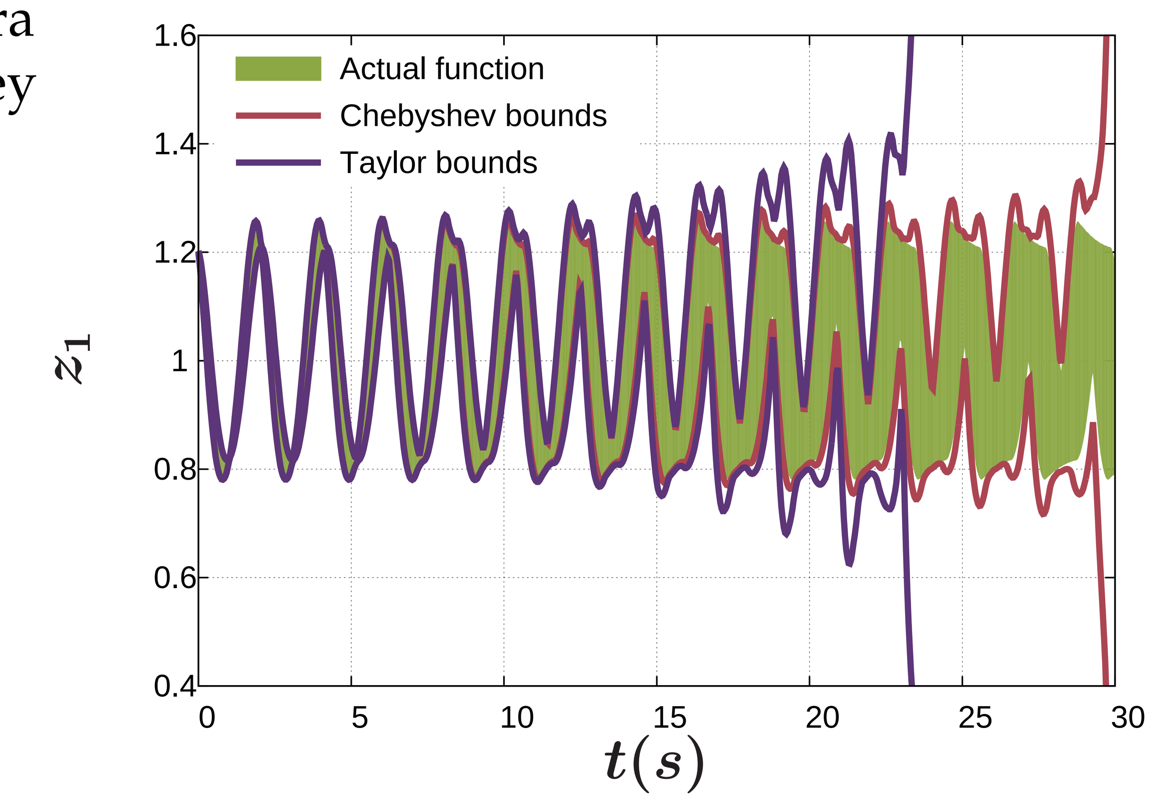
$$\dot{z}_1 = p z_1 (1 - z_2)$$
$$\dot{z}_2 = p z_2 (z_1 - 1)$$

With initial conditions:

$$z_1(0) = 1.2$$
$$z_2(0) = 1.1$$

Uncertain parameter:

$$2.95 \leq p \leq 3.05$$




- Bounds explode in both cases as the error propagation causes overestimation.
- Integration is done using a explicit iterative method which involves the discretisation of time.
- This involves a large number of factorable operations.
- The benefit of Chebyshev is compounded due to the number of operations.

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